IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

# The quantisation and measurement of momentum observables. II

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1981 J. Phys. A: Math. Gen. 14 2595

(http://iopscience.iop.org/0305-4470/14/10/015)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 129.252.86.83
The article was downloaded on 30/05/2010 at 17:46

Please note that terms and conditions apply.

J. Phys. A: Math. Gen. 14 (1981) 2595–2607. Printed in Great Britain

# The quantisation and measurement of momentum observables: II

Kay-Kong Wan† and Keith McFarlane‡

† Department of Theoretical Physics, University of St Andrews, St Andrews, Fife, Scotland
‡ School of Theoretical Physics, Dublin Institute for Advanced Studies, Dublin 4, Republic of Ireland

Received 4 December 1980

**Abstract.** Concepts of classical and quantum global measurability introduced in an earlier paper are discussed in greater detail and in the context of an $n$-dimensional Riemannian manifold. The ideas are illustrated with diagrams and examples. A notion of exact measurability is introduced and is shown to imply quantum global measurability. The physically important Killing momenta are shown to be exactly, and hence quantum globally, measurable.

## 1. Introduction

This paper is a sequel to and a development of a recent paper (Wan and McFarlane 1980, hereafter referred to as paper I) in which we introduced the concepts of classical and quantum global measurability and motivated their study by showing that measurability, in either of the above senses, requires of a momentum that it be complete, and therefore that it be quantisable by means of geometric procedures (Mackey 1963, Wan and Viazminsky 1977, 1979). The present paper is concerned with the concepts of classical and quantum global measurability in the context of an $n$-dimensional Riemannian manifold, their development, definition, and illustration by means of examples. A refinement of classical global measurability, termed exact measurability, is then introduced, shown to imply quantum global measurability, and moreover to be a property of the momenta associated with the physically important Killing vector fields.

## 2. Classical momentum measurement

The goal of this section is to construct the concept of classical global measurability in the context of a Riemannian manifold, and to relate it to the completeness of a momentum.

### 2.1. *The measurement process*

The problem here lies in the measurement of a momentum $P$ corresponding to a vector field $X$ on a complete Riemannian manifold $M$ of metric $G$. Around every non-critical point of $X$ there exists a local chart $x^i$ in terms of which $X$ assumes the form $\partial/\partial x^1$. In this chart $P$ becomes the momentum $p_1$ conjugate to $x^1$ and the metric assumes the usual tensor form $g^{ij}(x^k)$. In paper I we used the model of Aharanov and Safko (1975) to measure momenta and we shall use this model here. The model involves a collision between two particles, one a test particle described by unprimed quantities, and another a reference (measuring) particle described by primed quantities. The Hamiltonian describing the collision is given by

$$H = \frac{1}{2m} g^{ij}(x) p_i p_j + \frac{1}{2m'} g^{ij}(x') p'_i p'_j + \omega(t) g^{ij'}(x, x') p_i p'_{j'} \tag{1}$$

where $g^{ij'}(x, x')$ is the parallel propagator, and $\omega(t)$ is a function of $t$ satisfying

$$\omega(t) = \begin{cases} \omega_0, & t \in (0, T) \\ 0 & t \notin (0, T). \end{cases} \tag{2}$$

The non-uniqueness of the parallel propagator introduced by Synge (1971) causes no difficulty as we shall consider only local motion. Local motion is most simply described in terms of a local chart. In particular we can, in the neighbourhood of the collision, elect a local Cartesian chart $y^i$ such that (see paper I)

$$y^1 = b(m_0)(x^1 - a^1) \tag{3}$$

where $m_0$ is the point given by $x^i = a^i$, that is $m_0$ is the origin of the chart $y^i$. In terms of $y^i$ the Hamiltonian becomes

$$H = \frac{1}{2m} \eta^{ij} p_i^0 p_j^0 + \frac{1}{2m'} \eta^{ij} p_i'^0 p_j'^0 + \omega(t) \eta^{ij} p_i^0 p_j'^0 \tag{4}$$

where $\eta^{ij} = \delta^{ij}$ is the Cartesian metric, and $p_i^0$ is the momentum conjugate to $y^i$. Hamilton's equations of motion yield, upon taking the impulsive limit $\omega_0 T \mapsto \gamma$, that $\gamma p_i^0 = \Delta y'^i$, whence we may deduce the equation connecting $P$, the test particle's momentum, and $\Delta y'^1$ the measured displacement along $y^1$ as

$$P = p_1 = b(m_0) p_1^0 = \gamma^{-1} b(m_0)\, \Delta y'^1. \tag{5}$$

In the process the reference particle recoils by an amount and in a direction determined by the test particle's momentum. We can arrange the orientation of the reference particle prior to the interaction such that $p_i'^0 = 0$, $i \neq 1$, resulting in a recoil $\Delta y^1$ of the test particle along the $y^1$ axis, as is portrayed in figure 1. We shall adopt this option when employing the measuring procedure in the sequel. The parameter $\gamma$ is assumed known.

### 2.2. On local measurement

Let $A$ be an open neighbourhood of a point $m_0$ in the configuration manifold $M$ satisfying the following two conditions: (i) $A$ is coverable by a single local chart $x^i$ in terms of which the momentum $P$ becomes $p_1$ conjugate to $x^1$ and (ii) $A$ is sufficiently small that the departure within $A$ of $M$ from a Euclidean space is negligible. Then expressions (3) and (4) are well defined in $A$. We now introduce the notion of local measurement. A momentum measurement is said to be executed locally in $A$ if (i) no account is taken of position or momentum values without $A$, so that if a momentum value is to be determined within $A$ by means of the impulsive collision described above then neither the incident test particle, nor the reference particle employed, can so recoil

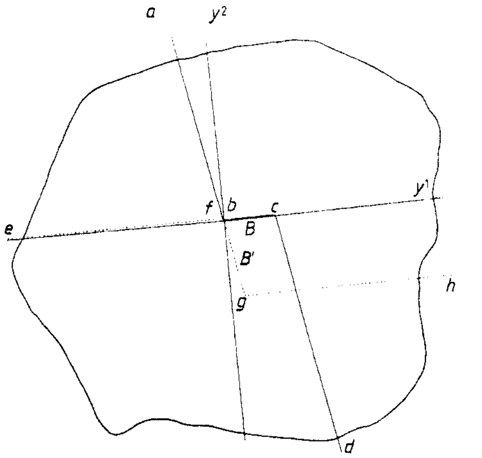


**Figure 1.** The geometry of the test/reference particle collision within the set $A$. The trajectory of the test particle of Cartesian momenta $p_i^0$ is the arc *abcd*, the recoil set $B$ demarking the extent of the uncertainty in position of the test particle during the collision; and the trajectory of the reference particle, aligned with the local flow direction *efc* and bearing Cartesian momentum $p_1'^0$ is similarly *efgh*, the local set $B'$ marking the recoil displacement of the reference particle during the collision.

as to leave $A$, and (ii) $A$ must possess a certain physical size, that is it must contain an open sphere of some fixed radius $d_0$, in order to be able to measure momenta corresponding to a given range of particle energies $E \leqslant E_A$. If the second condition is not satisfied then, for certain angles of incidence of the test particle, the reference particle will, after collision, so recoil as to leave $A$.

In executing a local measurement in $A$ we encounter two sources of possible error, one in the measurement of the reference particle recoil $\Delta y'^i$, and therefore in the evaluation of $p_1^0$, and the other in the identification of the point $m_0$ in $A$ from which $P$ may be calculated using (5). We shall assume that the former source of error may be made vanishingly small, or to be more definite we shall assume an exact measurement of $\Delta y'^1$ or, as is equivalent, an exact measurement of $p_1^0$. The latter source of error, however, has an intrinsic physical significance arising from the impulsive nature of the collision between test and reference particles which cannot be eliminated by perfection of the apparatus.

As depicted in figure 1 the test particle of momentum $P$ recoils within the one-dimensional set $B$ whose extent is determined solely by the parameter $\gamma$ and the reference particle momentum $p_1'^0$. It is this finite (non-zero) extent of $B$ which gives rise to an irreducible error; for it is impossible to say at which point $m$ of $B$ the momentum of the test particle was determined. We are faced, therefore, with the problem of defining the value attributed to $P$ by a locally executed measurement in $A$, and of quantifying the error arising from the variability of $b(m)$ within $B$. We shall take the value of $P$ as

$$P = b(B)p_1^0 \tag{6}$$

where

$$b(B) = \left(\int_B b(y^1)\,\mathrm{d}y^1\right)\left(\int_B \mathrm{d}y^1\right)^{-1}$$

which may be regarded as the mean value of $b(m)$ in $B$. For the error we shall take the standard deviation

$$\Delta_B P = \left(\int_B (p_1^0 b(y^1) - p_1^0 b(B))^2 \, dy^1\right)^{1/2} \left(\int_B dy^1\right)^{-1/2} = |p_1^0| \Delta_B b. \tag{7}$$

## 2.3. Global measurability and completeness

In paper I we introduced the concept of global measurability of $P$. We shall now generalise this notion explicitly to a Riemannian manifold. The aim is to establish a basically position-independent measuring process. The fundamental reason for this originates in quantum mechanical considerations, that is, the measurement of a quantised momentum should be basically position-independent so as not to conflict with the uncertainty principle. One then examines whether quantisable classical momenta possess this property, and whether it is related to the quantisability condition for momenta. Now in our model of measurement of classical momenta the measuring device is local, consisting of the reference particle (characterised by its momentum $p_1'^0$ conjugate to the local Cartesian coordinate $y'^1$) which interacts with the measured particle with a certain fixed value of $\gamma$. We shall call two measuring devices sited in the neighbourhoods of two distinct points $m_1$, $m_2$ in $M$ identical if their respective local properties at $m_1$ and $m_2$ are the same, that is, if they possess the same values of $p_1'^0$ and $\gamma$.

Returning to the momentum $P$ of the test particle, let $A_\Omega = \{A\}$ be an open covering of the maximal integral curve $\Omega$ of the vector field $X$ associated with $P$. Here every $A$ in $A_\Omega$ is assumed to satisfy the conditions described in § 2.2 so that a local measurement of $P$ may be performed in $A$. Local measurements of $P$ in $A$ using identical measuring devices produce a common value $\gamma p_1'^0$ for the recoil $\Delta y^1$ whatever the position of the test particle in $A_\Omega$ i.e. values independent of the choice of $A$†. In sharp contrast the error $\Delta_B p$ does depend on $A$. As we execute local measurements of $P$ in $A$, moving along $\Omega$ and using identical measuring devices, $\Delta_B P$ may vary unboundedly, depending upon the nature of the momentum involved. This would imply that we cannot meaningfully measure $P$ along $\Omega$ using identical measuring devices. In other words we cannot, in general, obtain a position-independent measuring process.

We are led to the notion of global measurability: a momentum $P$ is said to be classically globally measurable iff it is possible to elect, for each integral curve $\Omega$ of the vector field $X$ associated with $P$, a finite upper bound $\Delta_\Omega P$ to the set of errors arising from locally executed measurements in $A \in A_\Omega$ along $\Omega$ using identical measuring devices. This concept, illustrated in figure 2, is a generalisation of the idea of global measurability given in paper I‡. Mathematically all this, while clearly expressing physical intuition, is somewhat vague and ambiguous. It is necessary to give a precise mathematical definition. To do so we introduce families of local recoil sets. For each maximal integral curve $\Omega$ of $X$ let $\{B\}_{l_\Omega}$ be the family of all connected arcs $B$ of $\Omega$ of fixed metrical length $l_\Omega$. We call $\{B\}_{l_\Omega}$ a family of local recoil sets on $\Omega$. Note that $l_\Omega$, while having a fixed value for a given $\Omega$, can vary between integral curves. We can now give the desired definition: a momentum $P$ is classically globally measurable iff, for every integral curve $\Omega$ of $X$, it is possible to elect a finite upper bound $\Delta_\Omega P$ to the set of

† Here we assume that the Riemannian manifold is well behaved (has no singular point) so that we could have a minimal size for $A$, i.e. every $A$ contains an open sphere of some fixed common radius. This then allows the use of identical measuring devices along $A_\Omega$.

‡ Different measuring devices are allowed for different integral curves.

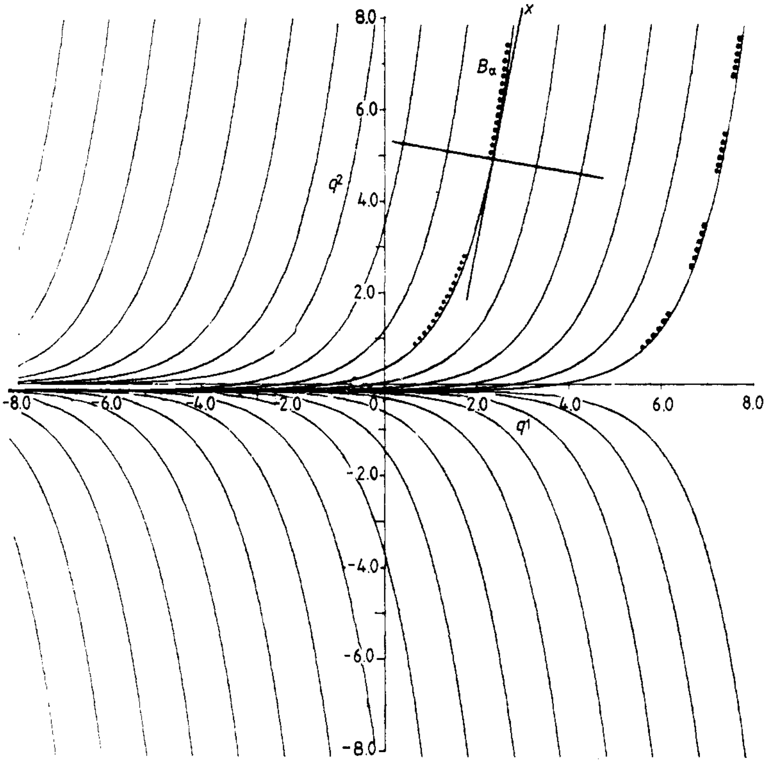


**Figure 2.** To illustrate the local recoil sets $B_\alpha$ defined on the integral curves of a vector field. The underlying graph is a family of maximal integral curves of the vector field $X = q^1(\partial/\partial q^1 + q^2\,\partial/\partial q^2)$ which has a set of critical points $q^1 = 0$ which separates the flow into two distinct regions $q^1 \gtrless 0$. We show along typical flow lines of the vector field a representative group of local recoil sets $B_\alpha$ together with the local Cartesian coordinate directions.

uncertainties $\Delta_B P$, $B \in \{B\}_{l_\Omega}$ (calculated with a preassigned value $p_0^1$ for all $B$) generated by the family of local recoil sets $\{B\}_{l_\Omega}$. This definition reflects our physical concept since for sufficiently small regions we can identify $\Delta y^1$ with the arc $B$ (hence the name local recoil set for $B$)†. The relevance of global measurability is seen in the following theorem whose proof is a simple extension of the proof for theorem 5 in paper I.

*Theorem 1.* On completeness and classical global measurability. A momentum $P$ is classically globally measurable only if it is complete.

## 3. Quantum momentum measurement

We address in this section the problem of the global measurability of a quantum momentum $Q(P)$, and shall begin by considering local measurement.

† The question arises as to whether such identification with a preassigned value of $\Delta y^1$ can be made everywhere along $\Omega$. We may indeed make such an identification along $\Omega$ provided that the curvature $K$ (Stoker 1969) of $\Omega$ is everywhere bounded. If there are regions in which $K$ is unbounded then the simple identification of $\Delta y^1$ and $B$ fails in these regions.

### *3.1. On local measurement*

Let $C_0^\infty(A)$ and $C_0^\infty(M)$ be the infinitely differentiable functions of compact support in $A$ and $M$ respectively. Let $Q_0(P)$ be the symmetric operator in $L^2(M)$ given by

$$Q_0(P) = -\mathrm{i}\hbar(X + \tfrac{1}{2}\operatorname{div} X) \tag{8}$$

with domain

$$DQ_0(P) = \{\psi \in L^2(M) | \psi \in C_0^\infty(M), Q_0(P)\psi \in L^2(M)\}.$$

Then the quantum momentum $Q(P)$, whenever it exists, is simply (see paper I) $Q_0^\dagger(P)$. Now let $x^i$ be a chart covering $A$ in terms of which $P = p_1$, the momentum conjugate to $x^1$. Naturally one tries to construct the quantum analogues, $Q(x^1)$ and $Q(p_1)$, of the classical canonical pair, $(x^1, p_1)$. However, neither $Q(x^1)$ nor $Q(p_1)$ can be simply defined as a self-adjoint operator in $L^2(M)$. The cause of the trouble is the fact that the chart $x^i$ is non-global in general, so that $x^1$ is not well defined without $A$. We should therefore seek operators in $L^2(M)$ with the property that their restrictions on $L^2(A)$ correspond to $x^1$ and $p_1$. Such operators may not be unique. This does not matter when one is dealing with problems purely in $A$ and $L^2(M)$ as we shall be in this section. A natural choice of these operators is as follows. For $Q(x^1)$ we define

$$Q(x^1)\psi = x^1\chi_A\psi \qquad \psi \in L^2(M) \tag{9}$$

where $\chi_A$ is the characteristic function of $A(\chi_A(m) = 1, m \in A$, and $\chi_A(m) = 0, m \notin A)$. For $Q(p_1)$ we take

$$Q(p_1) = Q(P) = Q_0^\dagger(P) \tag{10}$$

whenever $Q(P)$ exists. Confining ourselves to $C_0^\infty(A)$ we have the following well defined expressions:

$$Q(p_1)\varphi = Q_0(P)\varphi = -\mathrm{i}\hbar[\partial/\partial x^1 + \tfrac{1}{2}\partial(\ln\sqrt{g})/\partial x^1]\varphi \tag{11}$$

$$[Q(x^1), Q(p_1)]\varphi = [Q(x^1), Q_0(P)]\varphi = -\mathrm{i}\hbar\varphi \tag{12}$$

$$\Delta Q(x^1)\Delta Q(p_1) \geq \tfrac{1}{2}\hbar \qquad \Delta Q(x^1)\Delta Q_0(P) \geq \tfrac{1}{2}\hbar \tag{13}$$

where $\varphi \in C_0^\infty(A)$ and $\Delta Q = [\langle\varphi|Q^2|\varphi\rangle - \langle\varphi|Q|\varphi\rangle^2]^{1/2}$. Let $\Delta_A Q_0(P) = \inf(\Delta Q_0(P))$; then $\Delta_A Q_0(P)$ corresponds to $\sup Q(x^1)$ and to the maximum range of $x^1$ in $A$.

### *3.2. Global measurability and completeness*

Now let $A$ be an open neighbourhood of a point $m$ lying on a maximal integral curve $\Omega$ of $X$ associated with $P$. In $A$ the curve $\Omega$ is simply a coordinate curve along the coordinate $x^1$. We require $A$ to satisfy two additional conditions. (i) The arc $A \cap \Omega$ has a preassigned metric length $d_\Omega$. (ii) The maximum range of $x^1$ (generally different from arc length) in $A$ must equal the range of $x^1$ along the arc $A \cap \Omega$. We shall denote all such sets $A$ along $\Omega$ by $\{A\}_\Omega$ and call $\{A\}_\Omega$ a reference class of local sets constructed over $\Omega$. Note that $d_\Omega$ is the same for every $A \cap \Omega$, $A \in \{A\}_\Omega$. For different integral curves $d_\Omega$ may differ. We are now ready to introduce the notion of quantum global measurability.

*Definition.* A momentum $P$ is quantum globally measurable iff, for every maximal integral curve $\Omega$ of the associated vector field $X$ it is possible to elect a finite, upper bound $\Delta_\Omega Q_0(P)$ to the set of least uncertainties $\Delta_A Q_0(P)$ generated from a chosen reference class of local sets $A$ constructed over $\Omega$.

This definition is a natural generalisation of that given in our paper I, and enables by analogous arguments the deduction of the following theorem.

*Theorem 2.* On completeness and quantum global measurability. A momentum $P$ is quantum globally measurable only if it is complete.

*Proof.* Apply the argument of paper I for a similar theorem. Perhaps we should point out here that the notion of quantum global measurability is applicable whether $P$ is complete or not, since $Q_0(P)$ always exists, and that the least uncertainty $\Delta_A Q_0(P)$ in $A$ corresponds to the maximum range $\Delta x^1$ of $x^1$ in $A$.

## 4. On the relationship between classical and quantum global measurability

### 4.1. *Two illustrative examples on global measurability*

The first example is the familiar angular momentum $L_z$ on the Euclidean plane $\mathbb{R}^2$. In terms of global Cartesian coordinates $(q^1, q^2)$ on $\mathbb{R}^2$ and the corresponding chart $(q^1, q^2, p_1, p_2)$ on $T^*\mathbb{R}^2$, $L_z$ assumes the familiar form $L_z = q^1p_2 - q^2p_1$. The associated complete vector field is $X = q^1\partial/\partial q^2 - q^2\partial/\partial q^1$, whose maximal integral curves are given in terms of the flow $\Phi_t$: $\mathbb{R}^2 \mapsto \mathbb{R}^2$ by $\Phi_t(q^1, q^2) = (q^1 \cos t - q^2 \sin t,\ q^1 \sin t + q^2 \cos t)$. These integral curves form a family of concentric circles of centre (0, 0). The canonical coordinates in terms of which $X$ assumes the form $\partial/\partial\theta$ and the metric line element $ds^2 = (dq^1)^2 + (dq^2)^2$ the form $dr^2 + r^2\, d\theta^2$ are the (almost global) plane polar coordinates $(r, \theta)$ defined in the usual manner.

We now determine whether $L_z$ is classically globally measurable. Let us choose, for a representative integral curve $\Omega_a = \{(r, \theta)|r = a > 0\}$, a family $\{B_\alpha\}_{l_\alpha}$ of local recoil sets $B_\alpha$ as the family of arcs $B_\alpha$ of $\Omega$ of length $l_\alpha \ll 2\pi a$ centred at $r = a$, $\theta = \alpha$. Along each arc $B_\alpha$ of $\Omega$ we may introduce a local Cartesian coordinate $y^1$ defined by $y^1 = a\theta$ in terms of which $X = a\partial/\partial y^1$. It is now immediate that the quantity $b(m)$, $m \in \Omega_a$, is simply $b(m) = a$, a constant independent of $m \in \Omega_a$. Equations (6) and (7) give the measured value of $L_z$ to be $L_z = ap_0^1$ with an error $\Delta_{B_\alpha} L_z = 0$. Consequently $L_z$ is classically globablly measurable. Furthermore it is possible to measure $L_z$ without error within each recoil set $B_\alpha$, a result which reflects the rather special character of the angular momentum.

Let us now consider whether $L_z$ is quantum globally measurable. Let $A_0$ be an open neighbourhood of $m_0$ on $\Omega_a$ defined by

$$A_0 = \{(r, \theta)|a - \varepsilon/2 < r < a + \varepsilon/2,\ \theta \in (0, \delta\theta)\}$$

and let $A_\alpha$, $\alpha \in [0, 2\pi)$, be the set obtained by rotating $A_0$ through the angle $\alpha$. We can choose $\{A_\alpha|\alpha \in [0, 2\pi)\}$ as a reference class $\{A_\alpha\}_{\Omega_a}$ as defined in § 3.2. An illustration of the situation is in figure 3. It is obvious by symmetry that $\Delta_{A_\alpha} Q_0(L_z)$ is the same for every $A_\alpha$ because every $\varphi_\alpha \in C_0^\infty(A_\alpha)$ corresponds to $\varphi_{\alpha'} \in C_0^\infty(A_{\alpha'})$ and vice versa by rotation. It is also obvious that $\Delta_{A_\alpha} Q_0(L_z) = \inf(\Delta Q_0(L_z))$ is bounded; we conclude that $L_z$ is quantum globally measurable. Notice that $\Delta_{A_\alpha} Q_0(L_z)$ is the same for every $A_\alpha$.

As the second example we consider the momentum $P = (q^1 - q^2)p_1 + (q^1 + q^2)p_2$ again on the Euclidean plane $\mathbb{R}^2$. The associated complete vector field is $X =$

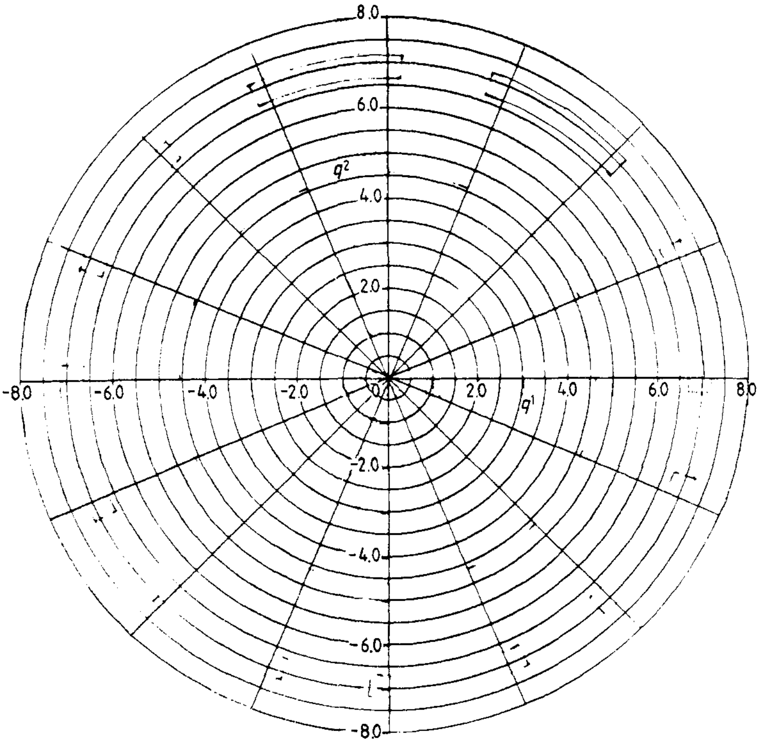


**Figure 3.** To illustrate the maximal integral curves of $L_z$ and of a representative grid of the polar coordinate system $(r, \theta)$. We also illustrate on the figure some typical local recoil sets $B_\alpha$ and some local sets $A_\alpha$.

$(q^1-q^2)\partial/\partial q^1+(q^1+q^2)\partial/\partial q^2$ whose maximal integral curves are given in terms of the flow map $\Phi_t$: $\mathbb{R}^2 \mapsto \mathbb{R}^2$ by

$$\Phi_t(q^1, q^2) = e^t(q^1 \cos t - q^2 \sin t, q^1 \sin t + q^2 \cos t).$$

Geometrically these are spirals radiating from the critical point at the origin. A set of coordinates $(x^1, x^2)$ in terms of which $X$ assumes the form $\partial/\partial x^1$ and the metric line element $ds^2$ the form $ds^2 = 2\exp[2(x^1+x^2)][(dx^1)^2+(dx^2)^2]$ may be defined using plane polar coordinates by

$$x^1 = \tfrac{1}{2}(\ln r + \theta) \qquad x^2 = \tfrac{1}{2}(\ln r - \theta).$$

Some integral curves of $X$ and a representative grid of coordinates $(x^1, x^2)$ are displayed in figure 4. Given a maximal integral curve $\Omega_0 = \{(x^1, x^2)|x^2 = x_0^2\}$, we can introduce a parameter $s(x^1)$ by

$$s(x^1) = \int_{-\infty}^{x^1} 2^{1/2} \exp(x^1 + x_0^2)\, dx^1 = 2^{1/2} \exp(x^1 + x_0^2) \tag{14}$$

which measures the metrical arc length from the critical point of $X$ to the point $(x^1, x_0^2)$ on $\Omega_0$ and in terms of which the quantity $b(m_0)$ along $\Omega_0$ takes the value $b(m_0) = \partial s/\partial x^1 = 2^{1/2}s$.

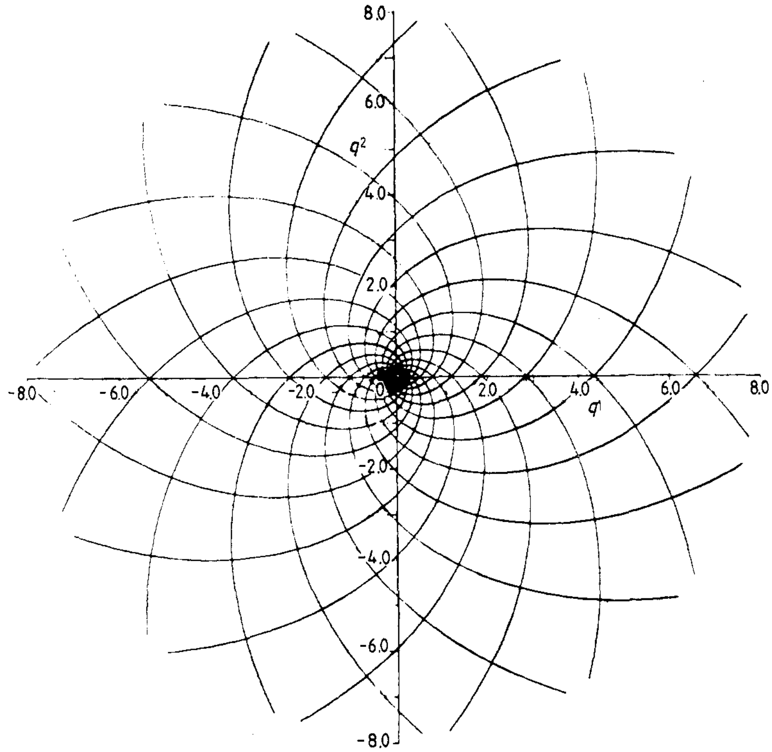


**Figure 4.** To illustrate the maximal integral curves of the vector field $X = (q^1 - q^2)\partial/\partial q^1 + (q^1 + q^2)\partial/\partial q^2$. The maximal integral curves of $X$ are the marked counter-clockwise spirals emanating from the origin of coordinates, the second family of curves being a representative set of coordinate arcs perpendicular to the integral curves of $X$.

Returning to the question of the classical global measurability of $P$, we choose as the family of local recoil sets the family $\{B_\alpha | \alpha \geqslant \frac{1}{2} d_0\}$ of arcs $B_\alpha$ of $\Omega_0$ each of metric length $d_0$ and centred at $s = \alpha$ on $\Omega_0$. Now equations (6) and (7) give the measured value of $P$ expected in $B_\alpha$ to be $P = b(B_\alpha)p_0^1$ with error $\Delta_{B_\alpha} P = 6^{-1/2} d_0 |p_0^1|$, which is independent of $\alpha$. Hence $P$ is classically globally measurable.

Finally, addressing the problem of the quantum global measurability of $P$, we construct on $\Omega_0$ a reference class $\{A_\alpha | \alpha \geqslant \frac{1}{2} d_0\}$ of local sets $A_\alpha$ with the property that $A_\alpha \cap \Omega_0 = B_\alpha$. It follows from equation (14) that the range of $x^1$ within $A_\alpha$ is $\ln[(2\alpha + d_0)/(2\alpha - d_0)]$ and hence the least uncertainty is given by

$$\Delta_{A_\alpha} Q_0(P) \geqslant \tfrac{1}{2}\hbar / \ln[(2\alpha + d_0)/(2\alpha - d_0)]$$

which tends to infinity as $\alpha$ tends to infinity†. As a result $P$ is not quantum globally measurable. This serves as a counter example to show that classical global measurability does not imply quantum global measurability. Furthermore, counter examples given in appendix 1 show that quantum global measurability does not imply classical global measurability either, and that the converses of both theorems 1 and 2 are false in general.

† The approach to the critical point as $q^1 \mapsto -\infty$ presents no problem since $\Delta_{A_\alpha} Q_0(P) \mapsto 0$ as $\alpha \mapsto \frac{1}{2} d_0$.

### 4.2. *Exact classical global measurability*

Having observed that neither all complete, nor all classical globally measurable, momenta are quantum globally measurable, we now enquire whether it is possible to identify a physically important subclass of the complete momenta which are both classically and quantum globally measurable. To this end we introduce a refinement of classical global measurability as follows.

*Definition.* A momentum $P$ is exactly classically globally measurable (exactly measurable) iff there exists a family $\{B\}_{I_\Omega}$ of local recoil sets for every maximal integral curve $\Omega$ of the associated vector field with the property that the error $\Delta_B P$ in every $B$ is zero.

This concept is motivated by recognition of the unique character of these momenta in being globally measurable without error. A prime example of these is the angular momentum $L_z$ studied in § 4.1. The relevance of exact measurability lies in the following three theorems.

*Theorem 3.* (see appendix 2) Practical criteria for exact measurability. Let $P = \xi^i(q)p_i$ be a complete momentum with the associated vector field $X = \xi^i(q)\partial/\partial q^i$. Then $P$ is exactly measurable iff either (1) for every maximal integral curve $\Omega = \Omega(s)$ parametrised by its arc length $s$, the quantity $b(m) = b(s)$ is constant along $\Omega(s)$ or (2) $\xi^i(q)$ satisfy the equation

$$\xi^i\xi^j(\xi_{i|j} + \xi_{j|i}) = 0$$

in every chart $q^i$ (the vertical bar denotes covariant differentiation).

*Theorem 4.* (see appendix 2) On exact measurability and quantum global measurability. If a momentum $P$ is exactly measurable then it is quantum globally measurable.

*Theorem 5.* (see appendix 2) On Killing momenta and exact measurability. Every complete Killing momentum $P$ defined by the requirement that its associated vector field $X$ is Killing is exactly measurable and therefore quantum globally measurable.

These theorems demonstrate the cohesion and physical relevance of the concepts of global measurability because these notions apply to the physically important Killing momenta. These theorems also tell us the property of the Killing momenta concerning measurement. Quantum mechanically an exactly measurable $P$ has the further feature that the error $\Delta_A Q_0(P)$ depends only on the length $d_0$ of $A$ and the particular integral curve $\Omega$ over which $A$ is defined, but does not depend on the exact location of $A$ on $\Omega$. In particular a Killing momentum like $L_z$ on $\mathbb{R}^2$ can be measured quantum mechanically along its integral curves with identical least error.

## 5. Conclusions

We have generalised the concepts of global measurability and introduced the notion of exact measurability. We have shown the existence of a large and physically important class of momenta, i.e. the Killing momenta, which are complete, hence quantisable, and are exactly measurable, hence classically and quantum globally measurable. We have also established the following relationships between completeness and measurabilities:

$$\{\text{exact measurability}\} \underset{\not\Leftarrow}{\Rightarrow} \{\text{classical global measurability}\}$$

$$\{\text{exact measurability}\} \underset{\not\Leftarrow}{\Rightarrow} \{\text{quantum global measurability}\}$$

$$\{\text{completeness}\} \underset{\not\Rightarrow}{\Leftarrow} \{\text{classical global measurability}\}$$

$$\{\text{completeness}\} \underset{\not\Rightarrow}{\Leftarrow} \{\text{quantum global measurability}\}$$

$$\{\text{classical global measurability}\} \underset{\not\Leftarrow}{\not\Rightarrow} \{\text{quantum global measurability}\}.$$

**Acknowledgment**

K McFarlane acknowledges the support of a Royal Society European Exchange Programme Fellowship, and the facilities made available to him by the Dublin Institute for Advanced Studies.

## Appendix 1. Counter examples

### *A1.1. The failure of the converse of theorems 1 and 2*

Consider, in the Euclidean space $\mathbb{R}$ with global Cartesian chart $y$, the momentum

$$P = \xi(y)p_y \qquad \xi(y) = y^3/(1+y^2\sin^2 y).$$

(i) *P is complete.* We have that

$$\begin{aligned}\int_{n\pi}^{(n+1)\pi} \xi^{-1}(y)\,\mathrm{d}y &= (n+\tfrac{1}{2})/[n^2(n+1)^2\pi^2] + \int_{n\pi}^{(n+1)\pi} y^{-1}\sin^2 y\,\mathrm{d}y \\ &\geqslant (n+\tfrac{1}{2})/[n^2(n+1)^2\pi^2] + (n+1)^{-1}\pi^{-1}\int_{n\pi}^{(n+1)\pi} \sin^2 y\,\mathrm{d}y \\ &= (n+\tfrac{1}{2})/[n^2(n+1)^2\pi^2] + \tfrac{1}{2}(n+1)^{-1}.\end{aligned}$$

Hence, $\forall y_0>0$ $\int_{y_0}^{y} \xi^{-1}(y)\,\mathrm{d}y \mapsto \infty$ as $y \mapsto \infty$. Also $\int_{y_0}^{y} \xi^{-1}(y)\,\mathrm{d}y \mapsto -\infty$ as $y \mapsto 0$. Similar results hold for $y_0<0$; hence $P$ is complete.

(ii) *Quantum global measurability.* For any interval $A=(y, y+d)$ of fixed length $d$, the range within $A$ of coordinate $x$ in which $\xi(y)\,\mathrm{d}/\mathrm{d}y$ goes to $\mathrm{d}/\mathrm{d}x$ is

$$\Delta x = \int_y^{y+d} \xi^{-1}(y)\,\mathrm{d}y \qquad y>0.$$

This is easily seen to tend to zero as $y \mapsto \infty$. Hence $\Delta_A Q(P)$ is unbounded and $P$ fails to be quantum globally measurable.

(iii) *Classical global measurability.* It will suffice to show that the errors $\Delta_B b$ generated by the class of sets $B_n = (n\pi, n\pi + d)$ for every fixed $d > 0$ increase without limit as $n$ tends to infinity. Setting $B_n = (\alpha, \beta)$ and considering the mean $b(B_n)$ we have that

$$b(B_n) = d^{-1} \int_\alpha^\beta \xi(y)\,\mathrm{d}y \leqslant d^{-1}\beta^3 \int_0^d (1+\alpha^2\sin^2 y)^{-1}\,\mathrm{d}y.$$

Therefore, upon noting the integrals

$$\int_0^{d<\pi/2} (1+\alpha^2\sin^2 y)^{-1}\,\mathrm{d}y = (1+\alpha^2)^{-1/2}\tan^{-1}[(1+\alpha^2)^{1/2}\tan d]$$

$$\int_0^{d<\pi/2} (1+\alpha^2\cos^2 y)^{-1}\,\mathrm{d}y = (1+\alpha^2)^{-1/2}\tan^{-1}[(1+\alpha^2)^{-1/2}\tan d]$$

we may deduce the upper bound

$$b(B_n) \leqslant \tfrac{1}{2}\beta^3 d^{-1}([2d/\pi] + \tfrac{1}{2})\pi(1+\alpha^2)^{-1/2}$$

in which $[2d/\pi]$ denotes the integer part of $2d/\pi$. To obtain a lower bound to the corresponding errors $\Delta_{B_n} b$, we consider the integral

$$\eta(B_n) = d^{-1}\int_\alpha^\beta \xi^2(y)\,\mathrm{d}y \geqslant d^{-1}\alpha^6 \int_0^d (1+\beta^2\sin^2 y)^{-2}\,\mathrm{d}y$$

which together with the identity

$$\int_0^{d<\pi/2} (1+\beta^2\sin^2 y)^{-2}\,\mathrm{d}y$$

$$= \frac{(1+\frac{1}{2}\beta^2)}{(1+\beta^2)^{3/2}}\tan^{-1}((1+\beta^2)^{1/2}\tan d) + \frac{\beta^2\sin d\cos d}{2(1+\beta^2)(1+\beta^2\sin^2 d)}$$

yields the bounds

$$\eta(B_n) \geqslant \begin{cases} \frac{1}{2}d^{-1}\alpha^6(1+\frac{1}{2}\beta^2)(1+\beta^2)^{-3/2}\pi & d \geqslant \pi/2 \\ d^{-1}\alpha^6(1+\frac{1}{2}\beta^2)(1+\beta^2)^{-3/2}\tan^{-1}[(1+\beta^2)^{1/2}\tan d] & d < \pi/2. \end{cases}$$

Finally observe that for sufficiently large $\beta$ or equivalently sufficiently large $n$, $(1+\beta^2)^{1/2}\tan d > 1$, and that therefore in all cases we have that for sufficiently large $n(d)$

$$\eta(B_n) \geqslant \tfrac{1}{4}d^{-1}\alpha^6(1+\tfrac{1}{2}\beta^2)(1+\beta^2)^{-3/2}\pi.$$

Finally, since $\Delta_{B_n} b = \eta(B_n) - b^2(B_n)$ we obtain the asymptotic lower bound $\Delta_{B_n} b \sim \pi\alpha^5/8d \sim \infty$, as was required.

### *A1.2. Quantum and classical global measurability*

Consider, again in $\mathbb{R}$, the momentum $P = \xi(y)p_y$, $\xi(y) = y^2/(1+y^4\sin^2 y)$. Using a procedure similar to § A1.1 above one can readily verify that this $P$ is complete and quantum globally measurable, but is not classically globally measurable.

## Appendix 2. On exact measurability

### *A2.1. Proof of theorem 3*

Part one is obviously true. So only part two needs proving. Let $x^i$ be a chart in terms of which $X = \xi^i \partial/\partial q^i = \partial/\partial x^1$. Let the metric in charts $q^i$ and $x^i$ be denoted respectively by $g_{ij}(q)$ and $\bar{g}_{ij}(x)$. Then $b^2(m) = \bar{g}_{11}(x)$ (see appendix 4 of paper I), or $b^2(m) = g_{ij}(\partial q^i/\partial x^1)(\partial q^j/\partial x^1) = g_{ij}\xi^i\xi^j$ (evaluated at $m \in \Omega$). Evaluating at $m \in \Omega$ and along $\Omega$ we have

$$\partial b/\partial s = 0 \Leftrightarrow \partial b/\partial x^1 = 0 \Leftrightarrow \xi^k \partial/\partial q^k (g_{ij}\xi^i\xi^j) = 0$$

the last expression leading directly to the required result.

### *A2.2. Proof of theorem 4*

$P$ being exactly measurable implies that $b(s) = b_0$, a constant by theorem 3. Using $b = \partial s/\partial x^1$, we have $dx^1 = b_0^{-1}\, ds$ which, after integrating, gives the maximum range $\Delta x^1 = b_0^{-1} \Delta s$. Since $\Delta s = d_0$ is the same for every local set $A$, $\Delta x^1$ is also the same for every $A$. Consequently $\Delta_A Q_0(P)$ the least uncertainty corresponding to the maximum range $\Delta x^1$ is the same for every $A$ resulting in the quantum global measurability of $P$. Notice also that $\Delta_A Q_0(P)$ is the same for every $A$.

### *A2.3. Proof of theorem 5*

This follows from theorem 3 since a Killing momentum $P = \xi^i p_i$ satisfies the Killing equations $\xi_{i|j} + \xi_{j|i} = 0$.

## References


Aharanov Y and Safko J L 1975 *Ann. Phys.* **91** 279

Mackey G W 1963 *The Mathematical Foundations of Quantum Mechanics* (New York: Benjamin) pp 100–4

Stoker J J 1969 *Differential Geometry* (New York; Wiley-Interscience) p 54

Synge J L 1971 *Relativity, the General Theory* (Amsterdam: North-Holland) p 59

Wan K-K and McFarlane K 1980 *J. Phys. A: Math. Gen.* **13** 2673

Wan K-K and Viazminsky C 1977 *Prog. Theor. Phys.* **58** 1030

—— 1979 *J. Phys. A: Math. Gen.* **12** 643